Math-601D-201: Lecture 20. Pseudo-convex domains with smooth boundary

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Pseudo-convex domains

$\Omega \subset \mathbb{C}^n$ connected open set.

Definition

Ω is a pseudo-convex domain iff for any compact set $K \subset \Omega$, the set

$$\hat{K}_{PSH(\Omega)} = \bigcap_{u \in PSH(\Omega)} \left\{ z \in \Omega,\ u(z) \leq \sup_K u \right\}$$

is compact in Ω.

Existence of smooth psh exhaustion functions

$\Omega \subset \mathbb{C}^n$ connected open set.

Theorem

Suppose Ω is pseudo-convex domain, $K \subset \Omega$ is compact, and let ω be an open neighborhood of the PSH-envelope $\hat{K}_{PSH(\Omega)}$.
Then there exists a smooth function $u \in \mathcal{C}^\infty(\Omega)$ such that

- *u is strictly psh, i.e. the complex Hessian*

$$\left[\frac{\partial^2 u}{\partial z_i \partial \bar{z}_j}\right]_{1 \leq i,j \leq n}$$

  is a positive definite hermitian form for all $z \in \Omega$;
- *$u < 0$ on K and $u > 0$ on $\Omega \setminus \omega$;*
- *$\{u < c\}$ is relatively compact for every $c \in \mathbb{R}$.*

Pseudo-convex domains with smooth boundary

$\Omega \subset \mathbb{C}^n$ connected open set with $\mathcal{C}^2$-boundary. Let $\rho \colon \mathbb{C}^n \to \mathbb{R}$ be a $\mathcal{C}^2$ function such that $\Omega = \{\rho < 0\}$ and $d\rho \neq 0$ on $\{\rho = 0\}$.

Theorem

Ω is pseudo-convex iff

$$\sum_{1 \leq i,j \leq n} \frac{\partial^2 \rho}{\partial z_i \partial \bar{z}_j} \lambda_i \bar{\lambda}_j \geq 0 \text{ if } \sum_{1 \leq i,j \leq n} \frac{\partial \rho}{\partial z_i} \lambda_i = 0 \ .$$

Levi problem

Theorem

Any pseudo-convex domain is a domain of holomorphy